AP CALCULUS AB AND BC

UNIT 8

Applications of Integration

AP EXAM WEIGHTING	10–15% AB 6–9% BC
CLASS PERIODS	~19–20 AB ~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 8

Multiple-choice: ~30 questions
Free-response: 3 questions

UNIT 8

AP EXAM WEIGHTING	10–15% AB	6–9% BC
CLASS PERIODS	~19–20 AB	~13–14 BC

Applications of Integration

BIG IDEA 1
Change CHA

- How is finding the number of visitors to a museum over an interval of time based on information about the rate of entry similar to finding the area of a region between a curve and the x-axis?

Developing Understanding

In this unit, students will learn how to find the average value of a function, model particle motion and net change, and determine areas, volumes, and lengths BC ONLY defined by the graphs of functions. Emphasis should be placed on developing an understanding of integration that can be transferred across these and many other applications. Understanding that the area, volume, and length BC ONLY problems studied in this unit are limiting cases of Riemann sums of rectangle areas, prism volumes, or segment lengths BC ONLY saves students from memorizing a long list of seemingly unrelated formulas and develops meaningful understanding of integration.

Building the Mathematical Practices

1.D 2.D 3.D 4.C

As in Unit 4, students will need to practice interpreting verbal scenarios, extracting relevant mathematical information, selecting an appropriate procedure, and then applying that procedure correctly and interpreting their solution in the context of the problem. Now that students have been exposed to application problems involving both differentiation and antidifferentiation, some may struggle to determine which procedure is applicable. Walk students through different types of scenarios and explain the underlying reasons why some situations call for differentiation while others call for integration.

This unit also involves geometric applications of integration. When using the disc and washer methods, focusing on orientation (i.e., horizontal or vertical) will help students determine whether the "thickness" is with respect to x or y. Students should practice solving variations on these calculus-based geometry problems until they can decide which variable to integrate with respect to without prompting. Relating graphical representations to symbolic representations, such as Riemann sums and definite integrals, develops these skills and helps students to master the content.

Preparing for the AP Exam

On the AP Exam, students need to identify relevant information conveyed in various representations. Key words, such as "accumulation" or "net change," help to identify mathematical structures and corresponding solution strategies. Some students confuse the average value and the average rate of change of a function on an interval. To alleviate confusion, first provide students with average value problems accompanied by relevant graphs and guide them to an understanding of why an average value may be less than, equal to, or greater than the midpoint of the range. Then review average rate of change problems from Unit 2 and present students with free-response questions that will allow them to practice distinguishing between average value and average rate of change problems.

In free-response questions, continue to require students to show supporting work by presenting a correct expression using appropriate notation and the mathematical structures of solutions, as in $V = \pi\int_1^4 \left[\left(f(x)-3 \right)^2 - \left(g(x)-3 \right)^2 \right] dx$, for example.

UNIT 8

Applications of Integration

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-4	**8.1 Finding the Average Value of a Function on an Interval**	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts	3.D Apply an appropriate mathematical definition, theorem, or test.	
CHA-5	**8.4 Finding the Area Between Curves Expressed as Functions of x**	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	8.5 Finding the Area Between Curves Expressed as Functions of y	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.6 Finding the Area Between Curves That Intersect at More Than Two Points	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	8.7 Volumes with Cross Sections: Squares and Rectangles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.8 Volumes with Cross Sections: Triangles and Semicircles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.9 Volume with Disc Method: Revolving Around the x- or y-Axis	3.D Apply an appropriate mathematical definition, theorem, or test.	

continued on next page

UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-5	**8.10 Volume with Disc Method: Revolving Around Other Axes**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-5	**8.11 Volume with Washer Method: Revolving Around the x- or y-Axis**	4.E Apply appropriate rounding procedures.	
CHA-5	**8.12 Volume with Washer Method: Revolving Around Other Axes**	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-6	**8.13 The Arc Length of a Smooth, Planar Curve and Distance Traveled** BC ONLY	3.D Apply an appropriate mathematical definition, theorem, or test.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 8. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	8.1	**Scavenger Hunt** Post around the room 8–10 problem cards, each of which also includes a solution to a previous problem. Include average value problems with tables and functions; also include tables that have values from 0 to 16, for example, but where the average value is only on the interval from 0 to 12. Card 1 could say: Find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$. A second card would have the answer to that card $\left(\frac{2}{\pi}\right)$ along with a new question: Find the average value of $f(x) = 3x^2 - 3$ on $[1, 3]$. A third card would have that answer (10) and so on. The answer to the last card goes on top of the first card.
2	8.6	**Round Table** In groups of four, each student has an identical paper with the same free-response question (e.g., **2015 AB #2(a)**), along with four labeled boxes representing steps in the problem: ▪ Identify all points of intersection. ▪ Set up the integral(s). ▪ Integrate by hand. ▪ Integrate using a calculator. Have students complete the first step on their paper, and then pass the paper clockwise to another member in their group. That student checks the first step and then completes the second step on the paper. Students rotate again and the process continues until each student has their own paper back.
3	8.9 8.10 8.11 8.12	**Quiz-Quiz-Trade** Create cards with problems revolving around the x- or y-axis and others revolving around other axes (e.g., $y = x$ or $y = 3$). Give each student a card and have them write their answer on the back. Students quiz a partner about their own card then switch cards and repeat the process with a new partner. For the first round, concentrate on just setting up the integrals (e.g., **2009 AB Form B #4(c)**, **2010 AB/BC #4(b)**, **2011 AB #3(c)**, and **2013 AB #5(b)**). In the second round, students can use their calculators to find the volume (e.g., **2001 AB #1(c)**, **2006 AB/BC #1(b)**, **2007 AB/BC #1(b)**, and **2008 AB Form B #1(b)**).

TOPIC 8.1
Finding the Average Value of a Function on an Interval

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.B

Determine the average value of a function using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-4.B.1

The average value of a continuous function f over an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCE

- Classroom Resource > **Motion**

TOPIC 8.2
Connecting Position, Velocity, and Acceleration of Functions Using Integrals

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.C

Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion.

ESSENTIAL KNOWLEDGE

CHA-4.C.1

For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.

TOPIC 8.3
Using Accumulation Functions and Definite Integrals in Applied Contexts

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.D

Interpret the meaning of a definite integral in accumulation problems.

CHA-4.E

Determine net change using definite integrals in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-4.D.1

A function defined as an integral represents an accumulation of a rate of change.

CHA-4.D.2

The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.

CHA-4.E.1

The definite integral can be used to express information about accumulation and net change in many applied contexts.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **Interpreting Context for Definite Integrals**

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 8.4

Finding the Area Between Curves Expressed as Functions of x

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.1

Areas of regions in the plane can be calculated with definite integrals.

TOPIC 8.5
Finding the Area Between Curves Expressed as Functions of y

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.2

Areas of regions in the plane can be calculated using functions of either x or y.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

TOPIC 8.6

Finding the Area Between Curves That Intersect at More Than Two Points

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.A

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.3

Areas of certain regions in the plane may be calculated using a sum of two or more definite integrals or by evaluating a definite integral of the absolute value of the difference of two functions.

TOPIC 8.7
Volumes with Cross Sections: Squares and Rectangles

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.1

Volumes of solids with square and rectangular cross sections can be found using definite integrals and the area formulas for these shapes.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

ILLUSTRATIVE EXAMPLES

- Illustrative examples of other cross sections in CHA-5.B.3:
- * The volume of a funnel whose cross sections are circles can be found using the area formula for a circle and definite integrals (see **2016 AB Exam FRQ #5(b)**).
- * The volume of a solid whose cross sectional area is defined using a function can be found using the known area function and a definite integral (see **2009 AB Exam FRQ #4(c)**).

TOPIC 8.8
Volumes with Cross Sections: Triangles and Semicircles

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.2

Volumes of solids with triangular cross sections can be found using definite integrals and the area formulas for these shapes.

CHA-5.B.3

Volumes of solids with semicircular and other geometrically defined cross sections can be found using definite integrals and the area formulas for these shapes.

TOPIC 8.9
Volume with Disc Method: Revolving Around the x- or y-Axis

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.1

Volumes of solids of revolution around the x- or y-axis may be found by using definite integrals with the disc method.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

TOPIC 8.10
Volume with Disc Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.2

Volumes of solids of revolution around any horizontal or vertical line in the plane may be found by using definite integrals with the disc method.

TOPIC 8.11
Volume with Washer Method: Revolving Around the x- or y-Axis

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.3

Volumes of solids of revolution around the x- or y-axis whose cross sections are ring shaped may be found using definite integrals with the washer method.

SUGGESTED SKILL

Communication and Notation

4.E

Apply appropriate rounding procedures.

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCE

- Classroom Resource > **Volumes of Solids of Revolution**

TOPIC 8.12

Volume with Washer Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.4

Volumes of solids of revolution around any horizontal or vertical line whose cross sections are ring shaped may be found using definite integrals with the washer method.

TOPIC 8.13
The Arc Length of a Smooth, Planar Curve and Distance Traveled BC ONLY

Required Course Content

ENDURING UNDERSTANDING

CHA-6

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE

CHA-6.A

Determine the length of a curve in the plane defined by a function, using a definite integral. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-6.A.1

The length of a planar curve defined by a function can be calculated using a definite integral. **BC ONLY**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.