AP CALCULUS AB AND BC

UNIT 6

Integration and Accumulation of Change

AP EXAM WEIGHTING	17–20% AB	17–20% BC
CLASS PERIODS	~18–20 AB	~15–16 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 6

Multiple-choice:
~25 questions (AB)
~35 questions (BC)
Free-response: 3 questions

UNIT 6

AP EXAM WEIGHTING	17–20% AB	17–20% BC
CLASS PERIODS	~18–20 AB	~15–16 BC

Integration and Accumulation of Change

BIG IDEA 1
Change **CHA**

- Given information about a rate of population growth over time, how can we determine how much the population changed over a given interval of time?

BIG IDEA 2
Limits **LIM**

- If compounding more often increases the amount in an account with a given rate of return and term, why doesn't compounding continuously result in an infinite account balance, all other things being equal?

BIG IDEA 3
Analysis of Functions **FUN**

- How is integrating to find areas related to differentiating to find slopes?

Developing Understanding

This unit establishes the relationship between differentiation and integration using the Fundamental Theorem of Calculus. Students begin by exploring the contextual meaning of areas of certain regions bounded by rate functions. Integration determines accumulation of change over an interval, just as differentiation determines instantaneous rate of change at a point. Students should understand that integration is a limiting case of a sum of products (areas) in the same way that differentiation is a limiting case of a quotient of differences (slopes). Future units will apply the idea of accumulation of change to a variety of realistic and geometric applications.

Building the Mathematical Practices

1.D 1.E 2.C 3.D

Students often struggle with the relationship between differentiation and integration. They think that integration is simply differentiation in reverse order. However, to apply the rules of integration correctly, students must think more strategically, taking into consideration how the "pieces" fit together. Students will need explicit guidance for choosing an appropriate antidifferentiation strategy that's based on the underlying patterns in different categories of integrands (e.g., using u-substitution when they recognize that the integrand is a factor of the derivative of a composite function or using integration by parts for an integrand, udv, that is related to a term in the derivative of the product uv **BC ONLY**).

Students also struggle with relating a symbolic limit of a Riemann sum to that limit expressed as a definite integral, because of the complexity of the expressions. To help students feel more comfortable working with these expressions, use explicit strategies, such as helping students to break complex expressions into familiar components, or matching expressions for a definite integral with the limit of a Riemann sum, and vice versa.

Preparing for the AP Exam

Students should be careful applying the chain rule, both when differentiating functions defined by integrals and when integrating using u-substitution. Students will need to recognize integrands that are factors of a chain rule derivative and should practice u-substitution until the process is internalized. Students will additionally need to recognize integrands that suggest strategies such as integration by parts or partial fractions and should use mixed practice in preparation for the exam **BC ONLY**.

When using a calculator to evaluate a definite integral in a free-response question, students should present the expression for the definite integral, including endpoints of integration, and an appropriately placed differential. When evaluating an integral without a calculator, students should present an appropriate antiderivative; they should include a constant of integration with indefinite integrals. As always, students should be careful about parentheses usage and should avoid writing strings of equal signs equating expressions that are not equal.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
CHA-4	**6.1 Exploring Accumulations of Change**	4.B Use appropriate units of measure.	
LIM-5	**6.2 Approximating Areas with Riemann Sums**	1.F Explain how an approximated value relates to the actual value.	
	6.3 Riemann Sums, Summation Notation, and Definite Integral Notation	2.C Identify a re-expression of mathematical information presented in a given representation.	
FUN-5	**6.4 The Fundamental Theorem of Calculus and Accumulation Functions**	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	6.5 Interpreting the Behavior of Accumulation Functions Involving Area	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-6	**6.6 Applying Properties of Definite Integrals**	3.D Apply an appropriate mathematical definition, theorem, or test.	
	6.7 The Fundamental Theorem of Calculus and Definite Integrals	3.D Apply an appropriate mathematical definition, theorem, or test.	
	6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y', and $\frac{dy}{dx}$).	
	6.9 Integrating Using Substitution	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	6.10 Integrating Functions Using Long Division and Completing the Square	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

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UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
FUN-6	**6.11 Integrating Using Integration by Parts** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	**6.12 Integrating Using Linear Partial Fractions** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
LIM-6	**6.13 Evaluating Improper Integrals** BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	**6.14 Selecting Techniques for Antidifferentiation**	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	

Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 6. Review the results in class to identify and address any student misunderstandings.

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	6.3	**Quickwrite** Present the class with several examples of definite integrals set equal to Riemann sums in summation notation, for example, $\int_{-2}^{5}(x^2+5)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{7}{n}\right)\left(\left(-2+\frac{7}{n}i\right)^2+5\right)$. Ask students to take five minutes to identify and write about all common elements between the two expressions and why they think the two expressions are equivalent. After finishing the five minutes, ask students to share their observations with the class.
2	6.9	**Look For a Pattern** Present students with several indefinite integrals and proposed, yet incorrect, antiderivatives, for example, $\int(5x+2)^{20}\,dx = \frac{1}{21}(5x+2)^{21}+C$. Ask them to check the antiderivatives by differentiating each and comparing to the original integrands. As students see that each antiderivative is incorrect, ask them to identify a pattern within the errors. Identifying this pattern will establish the foundation for integrating using substitution.
3	6.10	**Odd One Out** To help students select a strategy, form groups of four, presenting each student an indefinite integral whose integrand is rational. For each group, include one integrand that requires long division or completing the square. Ask students to decide if their example fits with the group. Identifying the odd one out will help students connect integrand form to the appropriate strategy.

TOPIC 6.1

Exploring Accumulations of Change

SUGGESTED SKILL

Communication and Notation

4.B

Use appropriate units of measure.

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.A

Interpret the meaning of areas associated with the graph of a rate of change in context.

ESSENTIAL KNOWLEDGE

CHA-4.A.1

The area of the region between the graph of a rate of change function and the x axis gives the accumulation of change.

CHA-4.A.2

In some cases, accumulation of change can be evaluated by using geometry.

CHA-4.A.3

If a rate of change is positive (negative) over an interval, then the accumulated change is positive (negative).

CHA-4.A.4

The unit for the area of a region defined by rate of change is the unit for the rate of change multiplied by the unit for the independent variable.

SUGGESTED SKILL

Implementing Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

AVAILABLE RESOURCE

- Classroom Resource > **Reasoning from Tabular Data**

TOPIC 6.2

Approximating Areas with Riemann Sums

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.A

Approximate a definite integral using geometric and numerical methods.

ESSENTIAL KNOWLEDGE

LIM-5.A.1

Definite integrals can be approximated for functions that are represented graphically, numerically, analytically, and verbally.

LIM-5.A.2

Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.

LIM-5.A.3

Definite integrals can be approximated using numerical methods, with or without technology.

LIM-5.A.4

Depending on the behavior of a function, it may be possible to determine whether an approximation for a definite integral is an underestimate or overestimate for the value of the definite integral.

TOPIC 6.3

Riemann Sums, Summation Notation, and Definite Integral Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.B

Interpret the limiting case of the Riemann sum as a definite integral.

ESSENTIAL KNOWLEDGE

LIM-5.B.1

The limit of an approximating Riemann sum can be interpreted as a definite integral.

LIM-5.B.2

A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition.

LEARNING OBJECTIVE

LIM-5.C

Represent the limiting case of the Riemann sum as a definite integral.

ESSENTIAL KNOWLEDGE

LIM-5.C.1

The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where n is the number of subintervals, Δx_i is the width of the ith subinterval, and x_i^* is a value in the ith subinterval.

LIM-5.C.2

A definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "Say" Some of the Notation**
- AP Online Teacher Community Discussion > **Definite Integral as the Limit of a Riemann Sum**

SUGGESTED SKILL

Implementing Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

ILLUSTRATIVE EXAMPLES

For FUN-5.A.1:

$f(x) = \int_0^x e^{-t^2}\,dt.$

AVAILABLE RESOURCES

- Professional Development > **The Fundamental Theorem of Calculus**
- Classroom Resource > **Functions Defined by Integrals**

TOPIC 6.4
The Fundamental Theorem of Calculus and Accumulation Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.1

The definite integral can be used to define new functions.

FUN-5.A.2

If f is a continuous function on an interval containing a, then $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$, where x is in the interval.

TOPIC 6.5

Interpreting the Behavior of Accumulation Functions Involving Area

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.3

Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as $g(x) = \int_a^x f(t)\,dt$.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 6.6

Applying Properties of Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.A

Calculate a definite integral using areas and properties of definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.A.1

In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.

FUN-6.A.2

Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.

FUN-6.A.3

The definition of the definite integral may be extended to functions with removable or jump discontinuities.

TOPIC 6.7

The Fundamental Theorem of Calculus and Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.B

Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

ESSENTIAL KNOWLEDGE

FUN-6.B.1

An antiderivative of a function f is a function g whose derivative is f.

FUN-6.B.2

If a function f is continuous on an interval containing a, the function defined by $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f for x in the interval.

FUN-6.B.3

If f is continuous on the interval $[a, b]$ and F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCE

- Professional Development > **The Fundamental Theorem of Calculus**

SUGGESTED SKILL

Communication and Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 6.8

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.C

Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

ESSENTIAL KNOWLEDGE

FUN-6.C.1

$\int f(x)dx$ is an indefinite integral of the function f and can be expressed as $\int f(x)dx = F(x)+C$, where $F'(x) = f(x)$ and C is any constant.

FUN-6.C.2

Differentiation rules provide the foundation for finding antiderivatives.

FUN-6.C.3

Many functions do not have closed-form antiderivatives.

TOPIC 6.9

Integrating Using Substitution

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.1

Substitution of variables is a technique for finding antiderivatives.

FUN-6.D.2

For a definite integral, substitution of variables requires corresponding changes to the limits of integration.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- Professional Development > **Applying Procedures for Integration by Substitution**
- AP Online Teacher Community Discussion > **U-Substitution with Improper Integrals**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E
Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.10

Integrating Functions Using Long Division and Completing the Square

Required Course Content

ENDURING UNDERSTANDING

FUN-6
Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D
For integrands requiring substitution or rearrangements into equivalent forms:

(a) Determine indefinite integrals.

(b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.3
Techniques for finding antiderivatives include rearrangements into equivalent forms, such as long division and completing the square.

TOPIC 6.11

Integrating Using Integration by Parts BC ONLY

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.E

For integrands requiring integration by parts:

(a) Determine indefinite integrals. **BC ONLY**

(b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-6.E.1

Integration by parts is a technique for finding antiderivatives. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E
Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.12
Integrating Using Linear Partial Fractions BC ONLY

Required Course Content

ENDURING UNDERSTANDING

FUN-6
Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.F
For integrands requiring integration by linear partial fractions:

(a) Determine indefinite integrals. **BC ONLY**

(b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-6.F.1
Some rational functions can be decomposed into sums of ratios of linear, nonrepeating factors to which basic integration techniques can be applied. **BC ONLY**

TOPIC 6.13

Evaluating Improper Integrals BC ONLY

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **U-Substitution with Improper Integrals**

Required Course Content

ENDURING UNDERSTANDING

LIM-6

The use of limits allows us to show that the areas of unbounded regions may be finite.

LEARNING OBJECTIVE

LIM-6.A

Evaluate an improper integral or determine that the integral diverges. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-6.A.1

An improper integral is an integral that has one or both limits infinite or has an integrand that is unbounded in the interval of integration. **BC ONLY**

LIM-6.A.2

Improper integrals can be determined using limits of definite integrals. **BC ONLY**

SUGGESTED SKILL

Implementing Mathematical Processes

1.C
Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 6.14

Selecting Techniques for Antidifferentiation

This topic is intended to focus on the skill of selecting an appropriate procedure for antidifferentiation. Students should be given opportunities to practice when and how to apply all learning objectives relating to antidifferentiation.