AP CALCULUS AB AND BC

# UNIT 5

# Analytical Applications of Differentiation

| | | |
|---|---|---|
| AP EXAM WEIGHTING | 15–18% AB<br>8–11% BC |
| CLASS PERIODS | ~15–16 AB<br>~10–11 BC |

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

## Personal Progress Check 5

**Multiple-choice: ~35 questions**
**Free-response: 3 questions**

UNIT 5

| | | |
|---|---|---|
| AP EXAM WEIGHTING | 15–18% AB | 8–11% BC |
| CLASS PERIODS | ~15–16 AB | ~10–11 BC |

# Analytical Applications of Differentiation

**BIG IDEA 3**
***Analysis of Functions*** FUN

- How might the Mean Value Theorem be used to justify a conclusion that you were speeding at some point on a certain stretch of highway, even without knowing the exact time you were speeding?
- What additional information is included in a sound mathematical argument about optimization that a simple description of an equivalent answer lacks?

## Developing Understanding

In this unit, the superficial details of contextual applications of differentiation are stripped away to focus on abstract structures and formal conclusions. Reasoning with definitions and theorems establishes that answers and conclusions are more than conjectures; they have been analytically determined. As when students showed supporting work for answers in previous units, students will learn to present justifications for their conclusions about the behavior of functions over certain intervals or the locations of extreme values or points of inflection. The unit concludes this study of differentiation by applying abstract reasoning skills to justify solutions for realistic optimization problems.

## Building the Mathematical Practices

1.E 2.D 2.E 3.E

The underlying processes of finding critical points and extrema are the foundation for the justifications students will write in this unit. Students should use calculators to graph a function and its derivatives to explore the related features of these graphs and confirm the results of their calculations.

Students often struggle with misinterpreting the characteristics of the graph of a derivative as though they are characteristics of the original function. Or, they use nonspecific language that conflates different functions (e.g., "it" rather than "$f''$"). To prevent ongoing misconceptions, hold students accountable for extreme precision by having them practice matching graphs of functions to their derivatives and requiring them to explain their reasons to a peer.

Students also tend to rely on insufficient evidence or descriptions in their justifications, stating, for example, that "the graph of $f$ is increasing because it's going up." This happens especially when examining derivative graphs on a calculator. Model calculus-based justifications (i.e., reasoning based on analysis of a derivative) both in discussion and in writing. Give students repeated opportunities to practice writing and revising their own justifications based on teacher feedback and feedback from their peers.

## Preparing for the AP Exam

Sound reasoning must be accompanied by clear communication on the AP Exam. It may be helpful for students to use the language in the question as a starting point. Suppose a question asks, "Does $g$ have a relative minimum, a relative maximum, or neither at $x=10$? Justify your answer." A student who writes, "$g$ has neither a relative maximum nor a relative minimum at $x=10$, because . . . ," has begun well. Similarly, given a graph of the derivative, $f'$, of a function, $f$, it is safer and easier for students to make arguments about $f$ based directly on the graph of the derivative, as in, "$f$ is concave up on $a<x<b$ because the graph of $f'$ is increasing on $a<x<b$." Students should always refer to $f$, $f'$, and $f''$ by name, rather than by "it" or "the function," which may leave the reader unsure of their intended meaning.

UNIT 5

# Analytical Applications of Differentiation

## UNIT AT A GLANCE

| Enduring Understanding | Topic | Suggested Skills | Class Periods<br>~15–16 CLASS PERIODS (AB)<br>~10–11 CLASS PERIODS (BC) |
|---|---|---|---|
| FUN-1 | **5.1 Using the Mean Value Theorem** | 3.E Provide reasons or rationales for solutions and conclusions. | |
| FUN-1 | **5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points** | 3.E Provide reasons or rationales for solutions and conclusions. | |
| FUN-4 | **5.3 Determining Intervals on Which a Function is Increasing or Decreasing** | 2.E Describe the relationships among different representations of functions and their derivatives. | |
| FUN-4 | **5.4 Using the First Derivative Test to Determine Relative (Local) Extrema** | 3.D Apply an appropriate mathematical definition, theorem, or test. | |
| FUN-4 | **5.5 Using the Candidates Test to Determine Absolute (Global) Extrema** | 1.E Apply appropriate mathematical rules or procedures, with and without technology. | |
| FUN-4 | **5.6 Determining Concavity of Functions over Their Domains** | 2.E Describe the relationships among different representations of functions and their derivatives. | |
| FUN-4 | **5.7 Using the Second Derivative Test to Determine Extrema** | 3.D Apply an appropriate mathematical definition, theorem, or test. | |
| FUN-4 | **5.8 Sketching Graphs of Functions and Their Derivatives** | 2.D Identify how mathematical characteristics or properties of functions are related in different representations. | |
| FUN-4 | **5.9 Connecting a Function, Its First Derivative, and Its Second Derivative** | 2.D Identify how mathematical characteristics or properties of functions are related in different representations. | |
| FUN-4 | **5.10 Introduction to Optimization Problems** | 2.A Identify common underlying structures in problems involving different contextual situations. | |
| FUN-4 | **5.11 Solving Optimization Problems** | 3.F Explain the meaning of mathematical solutions in context. | |
| FUN-4 | **5.12 Exploring Behaviors of Implicit Relations** | 1.E Apply appropriate mathematical rules or procedures, with and without technology.<br>3.E Provide reasons or rationales for solutions and conclusions. | |
| | Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 5. Review the results in class to identify and address any student misunderstandings. | | |

# SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

| Activity | Topic | Suggested Activity |
|---|---|---|
| 1 | 5.3 | **Critique Reasoning**<br>Arrange students in groups of four to six, provide them with a function's derivative (e.g., $g'(x) = 5x + 3$), and ask them to determine if $g(x)$ is increasing or decreasing at a specific $x$-value, for example, $x = -3$. Ask students to share the reasoning for their conclusion with classmates in their group. Members of the group can then provide feedback and suggestions. |
| 2 | 5.4<br>5.7 | **Think-Pair-Share**<br>Provide students with a graph of $f'$ and a graph of $f''$. Ask them to identify relative extrema and practice writing justifications for relative extrema using the first or second derivative test. Once they've written their justification, ask them to pair with a partner and share their justifications. Students can then discuss similarities or differences in their justification wording. |
| 3 | 5.5 | **Create a Plan**<br>Provide students with a function represented analytically on a closed interval. Ask them to discuss and write $x$-values that are viable candidates for absolute extrema. Once they have established the viable candidates, ask them to design a method for analyzing the behavior of the function's graph at the candidates and for identifying the extrema. |
| 4 | 5.8<br>5.9 | **Predict and Confirm**<br>Provide students with the graph of a differentiable function, for example, $f(x) = x^3 - 4x^2 + 4x + 1$, but do not provide the rule for the function. Ask students to sketch a graph of the derivative of the function. Once students are done, reveal the rule for $f(x)$. Ask students to calculate $f'(x)$, and use technology to graph $f'(x)$ and compare it to their sketched graph. |

**SUGGESTED SKILL**

*Justification*

**3.E**

Provide reasons or rationales for solutions and conclusions.

**AVAILABLE RESOURCES**

- Classroom Resource > **Why We Use Theorem in Calculus**
- AP Online Teacher Community Discussion > **Mean Value Existence Theorem**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**

# TOPIC 5.1
# Using the Mean Value Theorem

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-1**

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

### LEARNING OBJECTIVE

**FUN-1.B**

Justify conclusions about functions by applying the Mean Value Theorem over an interval.

### ESSENTIAL KNOWLEDGE

**FUN-1.B.1**

If a function $f$ is continuous over the interval $[a, b]$ and differentiable over the interval $(a, b)$, then the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

# TOPIC 5.2
# Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

## Required Course Content

### ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

### LEARNING OBJECTIVE

FUN-1.C

Justify conclusions about functions by applying the Extreme Value Theorem.

### ESSENTIAL KNOWLEDGE

FUN-1.C.1

If a function $f$ is continuous over the interval $[a, b]$, then the Extreme Value Theorem guarantees that $f$ has at least one minimum value and at least one maximum value on $[a, b]$.

FUN-1.C.2

A point on a function where the first derivative equals zero or fails to exist is a critical point of the function.

FUN-1.C.3

All local (relative) extrema occur at critical points of a function, though not all critical points are local extrema.

**SUGGESTED SKILL**

*Justification*

3.E

Provide reasons or rationales for solutions and conclusions.

**AVAILABLE RESOURCES**

- Classroom Resource > **Why We Use Theorem in Calculus**
- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Professional Development > **Justifying Properties and Behaviors of Functions**
- Classroom Resource > **Extrema**
- **On the Role of Sign Charts in AP Calculus Exams**

**SUGGESTED SKILL**

*Connecting Representations*

**2.E**

Describe the relationships among different representations of functions and their derivatives.

**AVAILABLE RESOURCE**

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

## TOPIC 5.3

# Determining Intervals on Which a Function Is Increasing or Decreasing

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

**FUN-4.A.1**

The first derivative of a function can provide information about the function and its graph, including intervals where the function is increasing or decreasing.

# TOPIC 5.4
# Using the First Derivative Test to Determine Relative (Local) Extrema

## Required Course Content

### ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

FUN-4.A.2

The first derivative of a function can determine the location of relative (local) extrema of the function.

**SUGGESTED SKILL**

*Justification*

3.D

Apply an appropriate mathematical definition, theorem, or test.

**AVAILABLE RESOURCE**

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

**SUGGESTED SKILL**

*Implementing Mathematical Processes*

**1.E**

Apply appropriate mathematical rules or procedures, with and without technology.

**AVAILABLE RESOURCE**

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

# TOPIC 5.5
# Using the Candidates Test to Determine Absolute (Global) Extrema

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

**FUN-4.A.3**

Absolute (global) extrema of a function on a closed interval can only occur at critical points or at endpoints.

# TOPIC 5.6
# Determining Concavity of Functions over Their Domains

## Required Course Content

### ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

FUN-4.A.4

The graph of a function is concave up (down) on an open interval if the function's derivative is increasing (decreasing) on that interval.

FUN-4.A.5

The second derivative of a function provides information about the function and its graph, including intervals of upward or downward concavity.

FUN-4.A.6

The second derivative of a function may be used to locate points of inflection for the graph of the original function.

**SUGGESTED SKILL**

*Connecting Representations*

2.E

Describe the relationships among different representations of functions and their derivatives.

**AVAILABLE RESOURCE**

- AP Online Teacher Community Discussion > **Second Derivative Test Wording and Justifying Concavity Intervals**

**SUGGESTED SKILL**

*Justification*

**3.D**

Apply an appropriate mathematical definition, theorem, or test.

**AVAILABLE RESOURCE**

- The Exam > **Commentary on the Instructions for the Free Response Section of the AP Calculus Exams**
- **On the Role of Sign Charts in AP Calculus Exams**

# TOPIC 5.7
# Using the Second Derivative Test to Determine Extrema

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

**FUN-4.A.7**

The second derivative of a function may determine whether a critical point is the location of a relative (local) maximum or minimum.

**FUN-4.A.8**

When a continuous function has only one critical point on an interval on its domain and the critical point corresponds to a relative (local) extremum of the function on the interval, then that critical point also corresponds to the absolute (global) extremum of the function on the interval.

# TOPIC 5.8
# Sketching Graphs of Functions and Their Derivatives

**SUGGESTED SKILL**

*Connecting Representations*

**2.D**

Identify how mathematical characteristics or properties of functions are related in different representations.

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

**FUN-4.A.9**

Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.

**FUN-4.A.10**

Graphical, numerical, and analytical information from $f'$ and $f''$ can be used to predict and explain the behavior of $f$.

**SUGGESTED SKILL**

*Connecting Representations*

**2.D**

Identify how mathematical characteristics or properties of functions are related in different representations.

**AVAILABLE RESOURCE**

- Professional Development > **Justifying Properties and Behaviors of Functions**

# TOPIC 5.9
# Connecting a Function, Its First Derivative, and Its Second Derivative

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

### ESSENTIAL KNOWLEDGE

**FUN-4.A.11**

Key features of the graphs of $f$, $f'$, and $f''$ are related to one another.

# TOPIC 5.10
# Introduction to Optimization Problems

**SUGGESTED SKILL**

*Connecting Representations*

**2.A**

Identify common underlying structures in problems involving different contextual situations.

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.B**

Calculate minimum and maximum values in applied contexts or analysis of functions.

### ESSENTIAL KNOWLEDGE

**FUN-4.B.1**

The derivative can be used to solve optimization problems; that is, finding a minimum or maximum value of a function on a given interval.

**SUGGESTED SKILL**

*Justification*

**3.F**

Explain the meaning of mathematical solutions in context.

# TOPIC 5.11
# Solving Optimization Problems

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

**FUN-4.C**

Interpret minimum and maximum values calculated in applied contexts.

### ESSENTIAL KNOWLEDGE

**FUN-4.C.1**

Minimum and maximum values of a function take on specific meanings in applied contexts.

# TOPIC 5.12
# Exploring Behaviors of Implicit Relations

## Required Course Content

### ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

### LEARNING OBJECTIVE

FUN-4.D

Determine critical points of implicit relations.

FUN-4.E

Justify conclusions about the behavior of an implicitly defined function based on evidence from its derivatives.

### ESSENTIAL KNOWLEDGE

FUN-4.D.1

A point on an implicit relation where the first derivative equals zero or does not exist is a critical point of the function.

FUN-4.E.1

Applications of derivatives can be extended to implicitly defined functions.

FUN-4.E.2

Second derivatives involving implicit differentiation may be relations of $x$, $y$, and $\frac{dy}{dx}$.

**SUGGESTED SKILLS**

*Implementing Mathematical Processes*

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

*Justification*

3.E

Provide reasons or rationales for solutions and conclusions.