AP CALCULUS AB AND BC

UNIT 1

Limits and Continuity

AP EXAM WEIGHTING	10–12% AB 4–7% BC
CLASS PERIODS	~22–23 AB ~13–14 BC

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Personal Progress Check 1

Multiple-choice: ~45 questions
Free-response: 3 questions (partial)

UNIT 1

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CLASS PERIODS	~22–23 AB	~13–14 BC

Limits and Continuity

BIG IDEA 1
Change CHA
- Can change occur at an instant?

BIG IDEA 2
Limits LIM
- How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?

BIG IDEA 3
Analysis of Functions FUN
- How do we close loopholes so that a conclusion about a function is always true?

Developing Understanding

Limits introduce the subtle distinction between evaluating a function at a point and considering what value the function is approaching, if any, as x approaches a point. This distinction allows us to extend understanding of asymptotes and holes in graphs with formal definitions of continuity. Consider reviewing rational functions when introducing limits, rather than beginning the year with a full review of precalculus topics. Limits are the foundation for differentiation (Unit 2), integration (Unit 6), and infinite series (Unit 10) BC ONLY. They are the basis for important definitions and for theorems that are used to solve realistic problems involving change and to justify conclusions.

Building the Mathematical Practices

2.B 2.C 3.C 3.D

Mathematical information may be organized or presented graphically, numerically, analytically, or verbally. Mathematicians must be able to communicate effectively in all of these contexts and transition seamlessly from one representation to another. Limits lay the groundwork for students' ongoing development of skills associated with taking what is presented in a table, an equation, or a sentence and translating that information into a graph (or vice versa). Help students explicitly practice matching different representations that show the same information, focusing on building their comfort level with translating analytical and verbal representations. This will be instrumental to their development of proficiency in this practice. The use of graphing calculators to help students explore these connections is strongly encouraged.

Mathematicians also explain reasoning and justify conclusions using definitions, theorems, and tests. A common student misunderstanding is that they don't need to write relevant given information before drawing the conclusion of a theorem. In Unit 1, students should be given explicit instruction and time to practice "connecting the dots" by first demonstrating that all conditions or hypotheses have been met and then drawing the conclusion.

Preparing for the AP Exam

This course is a full-year experience building toward mastery assessed using the AP Exam. Therefore, it is important to consider both specific content and skills related to each unit and to build a coherent understanding of the whole. After studying Unit 1, students should be prepared to evaluate or estimate limits presented graphically, numerically, analytically, or verbally. To avoid missed opportunities to earn points on the AP Exam, students should consistently practice using correct mathematical notation and presenting set-ups and appropriately rounded answers when using a calculator. Two sections of the exam do not allow calculator use. Some questions on the other two sections require it. From the first unit onward, emphasize the importance of hypotheses for theorems. Explore why each hypothesis is needed in order to ensure that the conclusion follows. Students should establish the practice of explicitly verifying that a theorem's hypotheses are satisfied before applying the theorem.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-1	**1.1 Introducing Calculus: Can Change Occur at an Instant?**	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
LIM-1	**1.2 Defining Limits and Using Limit Notation**	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.3 Estimating Limit Values from Graphs	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.4 Estimating Limit Values from Tables	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	1.5 Determining Limits Using Algebraic Properties of Limits	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.6 Determining Limits Using Algebraic Manipulation	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.7 Selecting Procedures for Determining Limits	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	1.8 Determining Limits Using the Squeeze Theorem	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.9 Connecting Multiple Representations of Limits	2.C Identify a re-expression of mathematical information presented in a given representation.	

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UNIT AT A GLANCE *(cont'd)*

Enduring Understanding	Topic	Suggested Skills	Class Periods ~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
LIM-2	**1.10 Exploring Types of Discontinuities**	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	1.11 Defining Continuity at a Point	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.12 Confirming Continuity over an Interval	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.13 Removing Discontinuities	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.14 Connecting Infinite Limits and Vertical Asymptotes	3.D Apply an appropriate mathematical definition, theorem, or test.	
	1.15 Connecting Limits at Infinity and Horizontal Asymptotes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-1	**1.16 Working with the Intermediate Value Theorem (IVT)**	3.E Provide reasons or rationales for solutions or conclusions.	
	Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 1. Review the results in class to identify and address any student misunderstandings.		

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

Activity	Topic	Sample Activity
1	1.2	**Notation Read Aloud** Begin by writing a limit expression in analytical form (e.g., $\lim_{x \to 0^-} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	**Create Representations** Present students with a limit expression in analytical form (e.g., $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$), and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	**Work Backward** Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simplify using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	**Discussion Groups** Give each group of students a piecewise-defined function, a graph paper, and a list of x-values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x-value, and explain why. Ask students to take turns recording the group's conclusion for each x-value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	**Think Aloud** In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

TOPIC 1.1
Introducing Calculus: Can Change Occur at an Instant?

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-1

Calculus allows us to generalize knowledge about motion to diverse problems involving change.

LEARNING OBJECTIVE

CHA-1.A

Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

ESSENTIAL KNOWLEDGE

CHA-1.A.1

Calculus uses limits to understand and model dynamic change.

CHA-1.A.2

Because an average rate of change divides the change in one variable by the change in another, the average rate of change is undefined at a point where the change in the independent variable would be zero.

CHA-1.A.3

The limit concept allows us to define instantaneous rate of change in terms of average rates of change.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

AVAILABLE RESOURCES

- Professional Development > **Definite Integrals: Interpreting Notational Expressions**
- AP Online Teacher Community Discussion > **How to "say" some of the notation**

TOPIC 1.2
Defining Limits and Using Limit Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.A

Represent limits analytically using correct notation.

ESSENTIAL KNOWLEDGE

LIM-1.A.1

Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x \to c} f(x) = R$.

X EXCLUSION STATEMENT

The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.

LIM-1.B

Interpret limits expressed in analytic notation.

LIM-1.B.1

A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

TOPIC 1.3
Estimating Limit Values from Graphs

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.1

The concept of a limit includes one sided limits.

LIM-1.C.2

Graphical information about a function can be used to estimate limits.

LIM-1.C.3

Because of issues of scale, graphical representations of functions may miss important function behavior.

LIM-1.C.4

A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

ILLUSTRATIVE EXAMPLES

For LIM-1.C.4:

- $\lim_{x \to 0} \frac{1}{x^2} = \infty$
- $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.
- $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.
- $\lim_{x \to 0} \frac{1}{x}$ does not exist.

AVAILABLE RESOURCES

- AP Calculator Policy
- Classroom Resource > AP Calculus Use of Graphing Calculators
- Professional Development > Limits: Approximating Values and Functions
- Classroom Resource > Approximation

SUGGESTED SKILL

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**
- Classroom Resource > **Approximation**

TOPIC 1.4
Estimating Limit Values from Tables

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.5

Numerical information can be used to estimate limits.

TOPIC 1.5
Determining Limits Using Algebraic Properties of Limits

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.D

Determine the limits of functions using limit theorems.

ESSENTIAL KNOWLEDGE

LIM-1.D.1

One-sided limits can be determined analytically or graphically.

LIM-1.D.2

Limits of sums, differences, products, quotients, and composite functions can be found using limit theorems.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

ILLUSTRATIVE EXAMPLES

- Factoring and dividing common factors of rational functions
- Multiplying by an expression involving the conjugate of a sum or difference in order to simplify functions involving radicals
- Using alternate forms of trigonometric functions

TOPIC 1.6
Determining Limits Using Algebraic Manipulation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.1

It may be necessary or helpful to rearrange expressions into equivalent forms before evaluating limits.

TOPIC 1.7
Selecting Procedures for Determining Limits

This topic is intended to focus on the skill of selecting an appropriate procedure for determining limits. Students should be given opportunities to practice when and how to apply all learning objectives relating to determining limits.

SUGGESTED SKILL

Implementing Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

ILLUSTRATIVE EXAMPLES

The squeeze theorem can be used to show $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Limits Questions**

TOPIC 1.8
Determining Limits Using the Squeeze Theorem

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.2

The limit of a function may be found by using the squeeze theorem.

TOPIC 1.9
Connecting Multiple Representations of Limits

This topic is intended to focus on connecting representations. Students should be given opportunities to practice when and how to apply all learning objectives relating to limits and translating mathematical information from a single representation or across multiple representations.

SUGGESTED SKILL

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- **AP Calculator Policy**
- Classroom Resource > **AP Calculus Use of Graphing Calculators**
- Professional Development > **Limits: Approximating Values and Functions**

SUGGESTED SKILL

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- AP Calculator Policy
- Classroom Resource > AP Calculus Use of Graphing Calculators

TOPIC 1.10
Exploring Types of Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.1

Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.

TOPIC 1.11
Defining Continuity at a Point

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.2

A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$.

SUGGESTED SKILL

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- AP Online Teacher Community Discussion > **Video on Continuity**

TOPIC 1.12
Confirming Continuity over an Interval

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.B

Determine intervals over which a function is continuous.

ESSENTIAL KNOWLEDGE

LIM-2.B.1

A function is continuous on an interval if the function is continuous at each point in the interval.

LIM-2.B.2

Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous on all points in their domains.

TOPIC 1.13
Removing Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.C

Determine values of x or solve for parameters that make discontinuous functions continuous, if possible.

ESSENTIAL KNOWLEDGE

LIM-2.C.1

If the limit of a function exists at a discontinuity in its graph, then it is possible to remove the discontinuity by defining or redefining the value of the function at that point, so it equals the value of the limit of the function as x approaches that point.

LIM-2.C.2

In order for a piecewise-defined function to be continuous at a boundary to the partition of its domain, the value of the expression defining the function on one side of the boundary must equal the value of the expression defining the other side of the boundary, as well as the value of the function at the boundary.

SUGGESTED SKILL

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCE

- The Exam > **2012 Exam, MCQ #9**

SUGGESTED SKILL

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 1.14
Connecting Infinite Limits and Vertical Asymptotes

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.1

The concept of a limit can be extended to include infinite limits.

LIM-2.D.2

Asymptotic and unbounded behavior of functions can be described and explained using limits.

TOPIC 1.15
Connecting Limits at Infinity and Horizontal Asymptotes

SUGGESTED SKILL

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.3

The concept of a limit can be extended to include limits at infinity.

LIM-2.D.4

Limits at infinity describe end behavior.

LIM-2.D.5

Relative magnitudes of functions and their rates of change can be compared using limits.

SUGGESTED SKILL

Justification

3.E

Provide reasons or rationales for solutions or conclusions.

AVAILABLE RESOURCES

- Professional Development > **Continuity and Differentiability: Establishing Conditions for Definitions and Theorems**
- Classroom Resource > **Why We Use Theorem in Calculus**

TOPIC 1.16
Working with the Intermediate Value Theorem (IVT)

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.A

Explain the behavior of a function on an interval using the Intermediate Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.A .1

If f is a continuous function on the closed interval $[a, b]$ and d is a number between $f(a)$ and $f(b)$, then the Intermediate Value Theorem guarantees that there is at least one number c between a and b, such that $f(c) = d$.